Superintegrability and higher order constants for classical and quantum systems

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We extend recent work by Tremblay, Turbiner, and Winternitz which analyzes an infinite family of solvable and integrable quantum systems in the plane, indexed by the positive parameter k. Key components of their analysis were to demonstrate that there are closed orbits in the corresponding classical system if k is rational, and for a number of examples there are generating quantum symmetries that are higher order differential operators than two. Indeed they conjectured that for a general class of potentials of this type, quantum constants of higher order should exist. We give credence to this conjecture by showing that for an even more general class of potentials in classical mechanics, there are higher order constants of the motion as polynomials in the momenta. Thus these systems are all superintegrable.

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I. INTRODUCTION

Recently Tremblay, Turbiner and Winternitz [1] studied a family of quantum mechanical systems in two dimensions with Hamiltonian

$$H = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + ar^2 + \frac{b}{r^2\cos^2 k\theta} + \frac{c}{r^2\sin^2 k\theta}. \tag{1}$$

They showed that for k an integer this system is superintegrable, as is the corresponding classical analog. We will prove that if k is rational then the corresponding classical problem

with Hamiltonian

$$H = p_r^2 + \frac{1}{r^2}p_\theta^2 + ar^2 + \frac{b}{r^2\cos^2 k\theta} + \frac{c}{r^2\sin^2 k\theta} \tag{2}$$

is also superintegrable. Indeed, it has two functionally independent constants of the motion that are polynomial in the momenta. One of these is always of second order corresponding to separation of variables in polar coordinates, viz

$$L_2 = p_\theta^2 + \frac{b}{\cos^2 k\theta} + \frac{c}{\sin^2 k\theta}. \tag{3}$$

It is the other generating constant of the motion on which we will concentrate.

To make this all explicit we consider the case $k = 2$ first and then give a general construction for rational k. Following this we consider a second potential

$$V = a\frac{(x+iy)^{k-1}}{(x-iy)^{k+1}} \tag{4}$$

where we also demonstrate superintegrability for k rational. These results strongly suggest that corresponding quantum systems are superintegrable.

II. THE POTENTIAL OF TREMBLAY,TURBINER AND WINTERNITZ

We look first at the $k = 2$ case. In Cartesian coordinates the classical Hamiltonian is

$$H = p_x^2 + p_y^2 + a(x^2+y^2) + b\frac{(x^2+y^2)}{(x^2-y^2)^2} + c\frac{(x^2+y^2)}{x^2y^2}. \tag{5}$$

There are two fundamental constants of the motion:

$$C_1 = (xp_y - yp_x)^2 + 4b\frac{x^2y^2}{(x^2-y^2)^2} + c\frac{(x^4+y^4)}{x^2y^2},$$

$$C_2 = (p_x^2 - p_y^2)^2 + [2ax^2 + 2b\frac{(x^2+y^2)}{(x^2-y^2)^2} - 2c\frac{(x^2-y^2)}{x^2y^2}]p_x^2$$
$$+[-4axy + 8b\frac{xy}{(x^2-y^2)^2}]p_xp_y + [2ay^2 + 2b\frac{(x^2+y^2)}{(x^2-y^2)^2} + 2c\frac{(x^2-y^2)}{x^2y^2}]p_y^2$$
$$+a^2(x^2-y^2)^2 + \frac{b^2}{(x^2-y^2)^2} + c^2\frac{(x^2-y^2)^2}{x^4y^4} + 8ab\frac{x^2y^2}{r(x^2-y^2)} + 2\frac{bc}{x^2y^2}.$$

The structure of the symmetry algebra is not worked out in [1]. We give the structure here. We set $R = \{C_1, C_2\}$. The Poisson algebra relations are

$$\{C_1, R\} = 32(H^2 - 2C_2)C_1 - 64(b+2c)C_2 + 64(b-c)H^2 - 128abC_1 - 128ab(b+2c)$$

$$\{C_2,R\} = 32C_2(C_2-H^2)+128aC_1H^2-384a^2C_1^2+128abC_2-64(b+4c)aH^2$$
$$+256a^2(2c-b)C_1+128a^2(b^2+40c^2+20bc).$$

There is a Casimir constraint

$$R^2 = 64C_1C_2(H^2-C_2)-64bH^4+128(b-c)C_2H^2-64(b+2c)C_2^2-128aC_1^2H^2+$$
$$256a^2C_1^3-256abC_1C_2+128a(b+4c)H^2C_1+256a^2(b-c)C_1^2-256ab(b+2c)C_2$$
$$+256a(7bc+b^2-2c^2)H^2-256a^2(b^2+4c^2+20bc)C_1-256a^2(2c+b)(b^2+16bc-4c^2).$$

From these relations we see that we have a closed Poisson algebra in the same sense as found for many well known superintegrable systems in two dimensions that are of second order [2, 3, 4]. In addition we can look for interesting one variable models of this algebra [5]. One such model is obtained by choosing $C_1 = c$. If we do this then

$$C_2 = \exp(\sqrt{-C-b-2c}\beta)+\frac{1}{2}E^2-2ab-\frac{E^2(4c-b)}{2(C+b+2c)}$$
$$-\frac{1}{16}\frac{(4Cc-4c^2-C^2+16bc)(4ab+9ac+4aC-E^2)^2}{(C+b+2c)^2}\exp(-\sqrt{-C-b-2c}\beta).$$

Here β is the variable conjugate to c.

For the quantum analogue of this system the Hamiltonian is

$$H = \partial_x^2+\partial_y^2+a(x^2+y^2)+b\frac{(x^2+y^2)}{(x^2-y^2)^2}+c\frac{(x^2+y^2)}{x^2y^2}$$

with quantum symmetries

$$C_1 = (x\partial_y-y\partial_x)^2+4b\frac{x^2y^2}{(x^2-y^2)^2}+c\frac{(x^4+y^4)}{x^2y^2},$$

$$C_2 = (\partial_x^2-\partial_y^2)^2+(2ax^2+2b\frac{(x^2+y^2)}{(x^2-y^2)^2}-2c\frac{(x^2-y^2)}{x^2y^2})\partial_x^2+$$
$$(-4axy+\frac{8bxy}{(x^2-y^2)^2})\partial_x\partial_y+(2ay^2+2b\frac{(x^2+y^2)}{(x^2-y^2)^2}+2c\frac{(x^2-y^2)}{x^2y^2})\partial_y^2$$
$$+(2ax-\frac{4c}{x^3})\partial_x+(2ay-\frac{4c}{y^3})\partial_y+a^2(x^2-y^2)^2+\frac{b^2}{(x^2-y^2)^2}+\frac{c^2{(x^2-y^2)}^2}{x^4y^4}+$$
$$8ab\frac{x^2y^2}{(x^2-y^2)^2}+\frac{2bc}{x^2y^2}+6c(\frac{1}{x^4}+\frac{1}{y^4}).$$

For these quantum operators there is corresponding closure given by the formulas ($\{\cdot,\cdot\}$, $\{\cdot,\cdot,\cdot\}$ are operator symmetrizers)

$$R = [C_1,C_2],$$

$$[C_1,R]=32C_1H^2-32\{C_1,C_2\}+64(b-c+2)H^2-64(b+2c+4)C_2-128a(b+1)C_1$$
$$-128a(b^2+2bc+4b+6c+4),$$
$$[C_2,R]=32C_2^2-32H^2C_2+128aC_1H^2+128a(b+1)C_2-64a(b+4c+6)H^2-384a^2C_1^2$$
$$-256a^2(b-2c-14)C_1+128a^2(-8+8c+18b+20bc+b^2+4c^2).$$

There is also the Casimir operator

$$R^2=32H^2\{C_1,C_2\}-\frac{32}{3}\{C_1,C_2,C_2\}-(64\beta+128\gamma+\frac{2816}{3})C_2^2+(128(\beta-\gamma)+\frac{2816}{3})H^2C_2$$
$$-(192+64\beta)H^4-128\alpha H^2C_1^2-128\alpha(\beta+1)\{C_1,C_2\}+\frac{128}{3}\alpha(12\gamma+3\beta+50)H^2C_1+256\alpha C_1^3$$
$$-\frac{256}{3}\alpha(44\beta+44+18\gamma+3\beta^2+6\beta\gamma)C_2-256\alpha^2(2\gamma-\beta46)C_1^2+\frac{256}{3}\alpha(42+22\gamma+40\beta$$
$$+3\beta^2-6\gamma^2+21\beta\gamma)H^2-\frac{256}{3}\alpha^2(152\gamma-88+182\beta+3\beta^2+12\gamma^2+60\beta\gamma)C_1$$
$$+\frac{256}{3}(280\gamma^2-80\beta^2+24\gamma^3+320\gamma+48\beta-4\beta\gamma-84\beta\gamma^3-54\gamma\beta^2-3\beta^2+28).$$

We now prove our central result that the classical Hamiltonian has, in addition to the obvious second order constant of the motion, another independent constant of the motion that is polynomial in the momenta. We use the results of a previous paper [6]. A similar approach was used by Verrier and Evans [7]. For the general potential we have in polar coordinates

$$V=\alpha r^2+\frac{\beta}{r^2\cos^2(k\theta)}+\frac{\gamma}{r^2\cos^2(k\theta)}.$$

In terms of the new variable $r=e^R$ the Hamiltonian assumes the form

$$H=e^{-2R}(p_R^2+p_\theta^2+\alpha e^{4R}+\frac{\beta}{\cos^2(k\theta)}+\frac{\gamma}{\sin^2(k\theta)}).$$

Applying the method of [6] to find the extra invariants we first need to construct a function $M(R,p_R)$ which satisfies $\{M,H\}=e^{-2R}$, or

$$(-4\alpha e^{4R}+2He^{2R})\partial_{p_R}M+2p_R\partial_RM=1.$$

This equation has a solution

$$M=\frac{i}{4\sqrt{L_2}}B$$

where

$$\sinh B=i\frac{(2L_2e^{-2R}-H)}{\sqrt{H^2-4\alpha L_2}},\quad \cosh B=\frac{2\sqrt{L_2}e^{-2R}p_R}{\sqrt{H^2-4\alpha L_2}},$$

and

$$L_2 = p_\theta^2 + \frac{\beta}{\cos^2(k\theta)} + \frac{\gamma}{\sin^2(k\theta)}$$

and we also have the relation (which we can use to consider M as a function of R alone):

$$p_R^2 + L_2 + \alpha e^{4R} - e^{2R} H = 0.$$

We now need to find the corresponding function $N(\theta, p_\theta)$ which satisfies $\{N, H\} = e^{-2R}$, or

$$(\frac{\beta}{\cos^2(k\theta)} + \frac{\gamma}{\sin^2(k\theta)})' \partial_{p_\theta} N - 2p_\theta \partial_\theta N = 1$$

where the prime denotes differentiation with respect to θ. This equation has a solution $N = -\frac{i}{4\sqrt{L_2}k} A$ where

$$\sinh A = i\frac{-\gamma + \beta - L_2\cos(2k\theta)}{\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}}, \quad \cosh A = \frac{\sqrt{L_2}\sin(2k\theta)p_\theta}{\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}}.$$

The constant of the motion is $M - N$, and, since it is constructed such that $\{M - N, L_2\} \neq 0$, it is functionally independent of L_2, [6].

From these expressions for M and N we see that if k is rational, $k = \frac{p}{q}$ (where p, q are relatively prime integers) then

$$\sinh(-4ip\sqrt{L_2}[N - M]) = -\sinh(qA + pB), \quad \cosh(-4ip\sqrt{L_2}[N - M]) = \cosh(qA + pB),$$

each give rise to a classical constant of the motion which is polynomial in the momenta.

This can be seen by observing that these constants of the motion can be expressed as factor functions of L_2 and H times a factor which is polynomial in the canonical momenta, via the relations

$$(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx, \ \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh(x+y) = \cosh x \sinh y + \sinh x \cosh y.$$

In particular,

$$\cosh nx = \sum_{j=0}^{[n/2]} \begin{pmatrix} n \\ 2j \end{pmatrix} \sinh^{2j} x \ \cosh^{n-2j} x,$$

$$\sinh nx = \sinh x \sum_{j=1}^{[(n=1)/2]} \begin{pmatrix} n \\ 2j-1 \end{pmatrix} \sinh^{2j-2} x \ \cosh^{n-2j-1} x.$$

Thus if p, q are both odd then

$$\cosh(qA + pB) = \frac{C}{[\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}]^q[\sqrt{H^2 - 4\alpha L_2}]^p},$$

$$\sinh(qA + pB) = \frac{\sqrt{L_2}\ D}{[\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}]^q[\sqrt{H^2 - 4\alpha L_2}]^p},$$

where C, D, are polynomial constants of the motion of orders $2(p+q), 2(p+q)-1$, respectively. If one of p, q is odd and the other even, then

$$\cosh(qA + pB) = \frac{\sqrt{L_2}\ C'}{[\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}]^q[\sqrt{H^2 - 4\alpha L_2}]^p},$$

$$\sinh(qA + pB) = \frac{D'}{[\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}]^q[\sqrt{H^2 - 4\alpha L_2}]^p},$$

where C', D', are polynomial constants of the motion of orders $2(p+q)-1, 2(p+q)$, respectively. (We have in fact produced two extra constants of the motion whose degree differs by 1. This is easily understood by realizing that we have one extra constant and its Poisson bracket with L_2.) For example, if $p = 1,\ q = 2$ we have (with $\mathcal{L} = -\gamma + \beta - L_2 \cos\theta$),

$$\cosh(2A + B) = \frac{2\sqrt{L_2}\left[(e^{-2R} p_R (L_2 \sin^2\theta p_\theta^2 - \mathcal{L}^2) - \sin\theta(2L_2 e^{-2R} - H)\mathcal{L}\right]}{[(L_2 - \beta - \gamma)^2 - 4\beta\gamma]\sqrt{H^2 - 4\alpha L_2}},$$

$$\sinh(2A + B) = \frac{\left[(2L_2 e^{-2R} - H)(L_2 \sin^2\theta p_\theta^2 - \mathcal{L}^2) + 4L_2 \sin\theta e^{-2R} p_\theta p_R \mathcal{L}\right]}{[(L_2 - \beta - \gamma)^2 - 4\beta\gamma]\sqrt{H^2 - 4\alpha L_2}}.$$

The bracketed quantities in the numerators are 5th and 6th order constants of the motion, respectively.

III. A NEW POTENTIAL

In addition to the potentials of Turbiner et. al. there is another family based on the same principle. To illustrate the properties of this family consider

$$H = p_x^2 + p_y^2 + a\frac{(x + iy)^6}{(x^2 + y^2)^4}. \tag{6}$$

This Hamiltonian admits three constants of the motion:

$$K_1 = (p_x - ip_y)^3 - \frac{a}{(x - iy)^3}[-(iy + 3x)p_x + (-ix + 3y)p_y],$$

$$K_2=(xp_y-yp_x)(p_x-ip_y)^3+\frac{a}{(x-iy)^3}[(3x^2+3ixy-2y^2)p_x^2-(2x^2+3ixy-3y^2)p_y^2-$$

$$i(x+3iy)(iy+3x)p_xp_y-a^2\frac{(x+iy)^3}{(x-iy)^6},$$

$$K_3=(xp_y-yp_x)^2+2ia\frac{y(3x^2-y^2)}{(x-iy)^3}.$$

The Poisson algebra relations are

$$\{K_1,K_2\}=3iK_1^2,\ \{K_1,K_3\}=6iK_2,$$

$$\{K_2,K_3\}=6iK_1(K_3+a),$$

together with the constraint

$$K_1^2K_3-K_2^2+a(K_1^2-H^3)=0.$$

There is also a sixth order symmetry K_1^2. There are a number of one variable models to consider for this Poisson algebra which help with the formulation of corresponding quantum problems viz.

$$(1):\ \ K_3=c,\ K_1=-\sqrt{\frac{aE^3}{a+c}}\cos(6\sqrt{c+a}\beta),\ K_2=-i\sqrt{aE^3}\sin(6\sqrt{c+a}\beta).$$

$$(2):\ \ K_1=c,\ K_2=3ic^2\beta,\ K_3=-8c^2\beta^2+\frac{aE^3}{c^2}-a.$$

$$(3):\ \ K_2=c,\ K_1=\frac{i}{3\beta},\ K_3=-9(c^2+aE^3)\beta^2-a.$$

We see that (1) indicates a realization of the quantum operators in terms of difference operators and (2) and (3) a realization in terms of differential operators.

Proceeding to the quantum analogue we obtain the operators

$$K_1=(\partial_x-i\partial_y)^3+\frac{a}{(x-iy)^3}[-(iy+3x)\partial_x+(3iy+x)\partial_y],$$

$$K_2=(x\partial_y-y\partial_x)(\partial_x-i\partial_y)^3+\frac{a}{(x-iy)^3}[i(2y^2-3ixy-3x^2)\partial_x^2-(3iy+x)(iy+3x)\partial_x\partial_y+$$

$$i(2x^2+3ixy-3y^2)\partial_y^2-2i(3iy+x)\partial_x-2(iy+3x)\partial_y-8i]+ia^2\frac{(x+iy)^3}{(x-iy)^6},$$

$$K_3=(x\partial_y-y\partial_x)^2+2iay\frac{(-y^2+3x^2)}{(x-iy)^3},$$

$$H=\partial_x^2+\partial_y^2+a\frac{(x+iy)^6}{(x^2+y^2)^4},$$

with the commutation relations

$$[K_1, K_2] = 3iK_1^2,\ [K_1, K_3] = 6iK_2 - 9K_1,$$

$$[K_2, K_3] = 3i\{K_1, K_2\} + i(27 + 6a)K_1 + 9K_2,$$

and the analogue of the constraint

$$\frac{1}{2}\{K_1, K_1, K_3\} - 3K_2^2 - i\frac{9}{2}\{K_1, K_2\} + (\frac{63}{2} + 3a)K_1^2 - 3aH^3 = 0.$$

A one dimensional model of this algebra is

$$K_1 = -\frac{i}{3x},\ K_2 = \frac{d}{dx}.$$

$$K_3 = -9x^2\frac{d^2}{dx^2} - 27x\frac{d}{dx} - (9 + a + 9aE^3x^2).$$

We now look at the question of what the constants of the motion might be for more general potentials of type (6). We consider the potentials

$$V = a\frac{(x + iy)^{k-1}}{(x - iy)^{k+1}}. \tag{7}$$

As in the previous example it is convenient to pass to variables R and θ. In these coordinates the Hamiltonian and the obvious constant of the motion L assume the form

$$H = \frac{(p_R^2 + p_\theta^2 + ae^{2ik\theta})}{e^{2R}}, \quad L = p_\theta^2 + ae^{2ik\theta}, \quad V = ae^{2ik\theta - 2R}$$

Using the the usual prescription for obtaining the extra constant we need to look for solutions of

$$2He^{2R}\partial_{p_R}M + 2p_R\partial_R M = 1.$$

and

$$-2iae^{2ik\theta}\partial_{p_\theta}N + 2p_\theta\partial_\theta N = 1$$

The new constant is then $M - N$. If k is an integer it is convenient to consider the solution such that $-ik\sqrt{L}(M - N) = A + kB$ where

$$\sinh A = \frac{p_\theta}{\sqrt{a}}e^{-ik\theta},\ \cosh A = \sqrt{\frac{L}{a}}e^{-ik\theta},$$

$$\sinh B = \frac{ip_R}{\sqrt{H}}e^{-R},\ \cosh B = \sqrt{\frac{L}{H}}e^{-R}.$$

If $k = \frac{p}{q}$ is rational then we consider $\sinh(qA + pB)$ and $\cosh(qA + pB)$ in order to obtain extra constants of the classical motion. For example, in the special case $k = 2$ we obtain

$$\sinh(A + 2B) = \frac{1}{\sqrt{a}H}(2(p_\theta + ip_R)L - p_\theta H e^{2R})e^{-2R}e^{-2i\theta},$$

$$\cosh(A + 2B) = \frac{\sqrt{L}}{\sqrt{a}H}(2L - He^{2R} + 2ip_\theta p_R)e^{-2R}e^{-2i\theta},$$

where we need only consider these hyperbolic functions multiplied by $\sqrt{a}H$ in the first case and $\sqrt{a}H/\sqrt{L}$ in the second, to obtain the polynomial solutions we seek.

IV. CONCLUSION

We have shown that for k rational all the classical mechanical systems (1) admit one second order constant of the motion as well as two others of higher order as polynomials in the momenta. This proves superintegrability and supports recent studies by Tremblay,Turbiner and Winternitz [1, 8] of the potentials with k rational where it has been demonstrated that all the orbits are closed. We also studied a new class of systems (7) and showed that again the systems are superintegrable and demonstrated how to find a maximal set of constants polynomial in the momenta. We provided some information about the structure of the symmetry algebras associated with all these systems.

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